The B.E. Journal of Theoretical Economics

Topics

Volume 10, *Issue* 1 2010 *Article* 37

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Recommended Citation

Mariano G. Runco (2010) "A Note on the Multidimensional Monopolist Problem and Intertemporal Price Discrimination," *The B.E. Journal of Theoretical Economics*: Vol. 10: Iss. 1 (Topics), Article 37.

Available at: http://www.bepress.com/bejte/vol10/iss1/art37

A Note on the Multidimensional Monopolist Problem and Intertemporal Price Discrimination*

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Abstract

This note analyzes a model of a monopolist selling multiple goods to a continuum of heterogeneous consumers. The implementation of Direct Revelation Mechanisms is analyzed in that setting, finding that it is possible for the monopolist to implement all Stochastic Incentive Compatible Mechanisms by committing to post a decreasing sequence of prices. The posted prices depend on time and have the desirable property of being step functions. When the optimal mechanisms are stochastic, it is optimal for the monopolist to price discriminate over time, contrary to the conventional wisdom that a single-good monopolist committed to an ex-ante price strategy will not price discriminate.

KEYWORDS: multidimensional mechanism design, stochastic mechanisms, intertemporal price discrimination, monopolist

*I would like to thank Hector Chade, Alejandro Manelli, Richard Rogerson, Edward Schlee, the editor and one referee for helpful advice. All remaining errors are mine.

1 Introduction

In this note we analyze the implementation of Stochastic Direct Revelation Mechanisms (SDRM). It is a well-known result that in a model of a monopolist selling one good to a consumer with private valuation in $[0,1]$, the optimal (revenue maximizing) mechanism is deterministic. When on the other hand a monopolist sells multiple goods, then revenue-maximizing direct mechanisms are often stochastic (see Manelli and Vincent (2007)). In other words, the probability that some consumers, upon reporting their valuation, obtain the given goods is strictly between zero and one. One trivial way to implement stochastic mechanisms in this setting is to use lotteries. The monopolist posts prices and if the consumer buys, a randomization device is used to determine whether the consumer takes the good or not.

Given this fact a natural question arises: Can we implement stochastic direct mechanisms in a deterministic manner? If so, what are the properties of the indirect mechanism? The answer provided in this note is that Intertemporal Price Discrimination (reducing the price of a good as time passes) implements Stochastic Direct Revelation Mechanisms. We prove that all Stochastic Direct Revelation Incentive Compatible Mechanisms can be implemented in a dynamic setting where a monopolist commits to post a sequence of decreasing prices. Therefore, if Stochastic Mechanisms are optimal, then price discounts are also optimal in the dynamic model, thus providing another rationale for IPD. Moreover we find that if the seller offers discounts, prices are step functions. For simplicity we consider the case of two goods. The results also hold for an N-dimensional monopolist. However, in that case notational complexities become a serious problem and the gain in understanding is marginal.

This note is motivated by Manelli and Vincent (2007). They analyze the problem for the multidimensional monopolist using a mechanism design approach, they restrict attention to Direct Revelation Mechanisms and find that optimal mechanisms *might* include randomization over the provision of goods. Unlike the single-dimensional problem, the characteristics of the optimal mechanism depend on the distribution of types. No mention is made of the issue of implementation of stochastic mechanisms except the straightforward use of lotteries. In fact, given that the optimality of stochastic mechanisms seems troublesome because the use of such lotteries is not observed in practice, they provide conditions in a different paper (Manelli and Vincent (2006)) such that deterministic direct revelation mechanisms are optimal.[1] We show in this paper that such concerns are unwarranted since stochastic mechanisms can also be easily implemented in a deterministic way.

[1] They state in the introduction "We focus on price schedules (or bundling) because we believe they are simple, easily implemented institutions, and are natural extensions of the optimal mechanism in the one good case".

The first model to deal with the problem of IPD was Stokey (1979). She developed a model of a committed monopoly selling a zero cost durable good to a continuum of potential buyers. She unexpectedly finds that the optimal strategy for the monopolist is to post a price and leave it unchanged. Later papers have analyzed models in which IPD is optimal for the monopolist. For example, Conlisk, Gerstner and Sobel (1984), Landsberger and Meilijson (1985), Rustichini and Villamil (1996,2000) and Koh (2006). Those models have included some sort of distortion or heterogeneity to obtain IPD. It is not surprising that this has been the case, because it is possible in the single-good case to derive the Direct Revelation Mechanism that maximizes monopolist expected profits. In the intertemporal durable-good setting this Direct Revelation Mechanism can be implemented by a posted constant price. Also, prices depend on the distribution of valuations, but not the optimal strategy.

2 Model

Consider a risk-neutral monopolist selling two goods to a consumer who has private information about her valuations. The consumer wants at most one unit of each good and her preferences are given by

$$U(q,\theta,t) = q\cdot\theta - t$$

where valuations $\theta \in [0,1]^2$, $q \in \{0,1\}^2$ depending on whether she buys or not and the monetary transfer $t \in R$. Valuations are distributed according to the atomless distribution $F(\theta)$ on the unit square.

The monopolist's objective is to design a direct revelation mechanism to sell goods so as to maximize expected profits. A Direct Revelation Mechanism is a pair of functions $p : [0,1]^2 \to [0,1]^2$ representing the probabilities that the buyer gets the goods and $t : [0,1]^2 \to R$ is the expected transfer from the buyer to the seller when she reports the vector θ.

The solution to this problem is very difficult to find. However, we are not interested in the optimal mechanism (or mechanisms) per se, but just in the property that optimal mechanisms *might* include randomization over the provision of goods. Unlike the single-dimensional problem, which was analyzed by Myerson (1981), characteristics of the optimal mechanism depend on the distribution of types. Therefore a richer set of mechanisms is optimal for the seller. In some cases, that is, for some $F(\theta)$, it may be optimal to separate consumers into extra market pieces in order to extract more surplus, as shown below.

DOI: 10.2202/1935-1704.1659

Example 1. *The example shows a mechanism with 6 different pieces.*

$$p(\theta)=\begin{cases}(1,1) & \text{if } \theta\in A_{(1,1)}\\ (1,0) & \text{if } \theta\in A_{(1,0)}\\ (.5,0) & \text{if } \theta\in A_{(.5,0)}\\ (0,1) & \text{if } \theta\in A_{(0,1)}\\ (0,.5) & \text{if } \theta\in A_{(0,.5)}\\ (0,0) & \text{if } \theta\in A_{(0,0)}\end{cases}\qquad t(\theta)=\begin{cases}1 & \text{if } \theta\in A_{(1,1)}\\ .7 & \text{if } \theta\in A_{(1,0)}\\ .3 & \text{if } \theta\in A_{(.5,0)}\\ .7 & \text{if } \theta\in A_{(0,1)}\\ .3 & \text{if } \theta\in A_{(0,.5)}\\ 0 & \text{if } \theta\in A_{(0,0)}\end{cases}$$

In this example there are six ”‘pieces”’. For instance, the consumer can get the bundle paying 1, she can pay .3 and get good 1 with probability $1/2$, she can get no good and pay nothing and so on. In the lottery the monopolist tosses a fair coin, and if heads comes up, the buyer receives good 1. On the other hand, if tails comes up, she receives nothing. However, in practice we do not observe mechanisms in which the monopolist chooses randomly whether to provide the good or not. We almost invariantly observe posted prices as selling devices.

Now we consider the indirect implementation of the Direct Revelation Mechanisms. Since we restrict attention to piecewise linear mechanisms, we can write them as follows:

$$(p,t)^i \qquad i=0,...,N.$$

This means that the mechanism divides the type space into N different market segments. All types in each market segment are treated equally (bunching). Notice that an optimal mechanism, given any prior $F(\theta)$, will always have at least two segments: one in which no good is ever trade $(p_1,p_2,t)=(0,0,0)$ and other in which all goods are traded with certainty $(p_1,p_2,t)=(1,1,t_B)$. As noted before, the optimal mechanism (p^*,t^*) (and therefore the number of market segments) will depend on the distribution of types $F(\theta)$.

We next describe the setting in which any DRM will be implemented. Consider a monopolist selling two indivisible goods to an infinitely-lived consumer. As usual, the monopolist does not know the type of the consumer but she knows the distribution $F(\theta)$. Both the monopolist and the consumer have discount rate $\delta\in(0,1)$. Time is a continuous variable.[2] Ex-ante, the monopolist commits to offering goods and/or the bundle at known prices. There are three potential kinds of offers: one involves the bundle, and one involves each good. Once a given offer is made, it remains binding until another offer of the same kind is made. If an offer is accepted by the buyer the process is over.

From the consumer's point of view, she sees a menu of binding offers at each moment in time. And more importantly, she knows the sequence of future

[2]This is a crucial assumption, in the sense that our results are not valid without it.

offers (what goods, what prices and when). Basically, her decision at time τ is to choose any active offer or wait for a better deal. From her point of view there is no uncertainty whatsoever.

We now define a map T that transforms probabilities of delivery into time. Define $\mathscr{D} = \{(p,t)^i \mid i = 0,....,N\}$. This set contains N + 1 pieces of the given IC DRM. Define also $\mathscr{M} = \{(\tau,\bar{t})^i \mid i = 0,....,N\}$ where $\tau \in [0,\infty]$ and $\bar{t} \in R$.

The function $T : \mathscr{D} \to \mathscr{M}$ is defined as follows

$$T((p,t)^i) = (\tau_1^i, \tau_2^i, \bar{t}^i)$$

where $\tau_1^i = \frac{\log(p_1^i)}{\log(\delta)}$, $\tau_2^i = \frac{\log(p_2^i)}{\log(\delta)}$.
$\bar{t}^i = \frac{t^i}{\max\{p_1^i, p_2^i\}}$ or $\bar{t}^i = 0$ if $p_1^i = p_2^i = 0$.

Before interpreting the new variables we need to define the following

$$\alpha = \begin{cases} 1 & \text{if } \tau_1^i \leq \tau_2^i \\ 2 & \text{if } \tau_1^i > \tau_2^i \end{cases} \qquad \beta = \begin{cases} 1 & \text{if } \tau_1^i > \tau_2^i \\ 2 & \text{if } \tau_1^i \leq \tau_2^i \end{cases}$$

The function $\min\{\tau_1^i, \tau_2^i\}$ gives the time when a payment is made and good α is delivered while $\max\{\tau_1^i, \tau_2^i\}$ is the time when good β is delivered.

Thus, the expected utility, at time zero, of a consumer with types (θ_1, θ_2) choosing any option $b \in \mathscr{M}$ is given by

$$v(\theta, b) = \delta^{\min\{\tau_1^i, \tau_2^i\}}(\theta_\alpha + \delta^{(\max\{\tau_1^i, \tau_2^i\} - \min\{\tau_1^i, \tau_2^i\})}\theta_\beta - \bar{t}^i)$$

Applying the function T to the earlier example we get, assuming $\delta = .9$, that $\mathscr{M} = ((0,0,1);(0,\infty,.7);(6.57,\infty,.6);(\infty,0,.7);(\infty,6.57,.6);(\infty,\infty,0))$.

As before, the buyer is shown the set $\mathscr{M}$ as a menu. Each element is a choice she can make. The $(0,0,1)$ option gives the consumer the bundle of goods 1 and 2 at time zero at a price of 1. The $(0,\infty,.7)$ option gives good 1 to the consumer at time 0 at a price of .7. The $(6.57,\infty,.6)$ option gives good 1 to the consumer at time 6.57 at a price of .6. The last option is simply the possibility of not buying anything (and of course not paying). Now if we look at the price of good 1 (or 2, they are symmetric) as a function of time we observe that $p_1 = .7$ for the interval $[0, 6.57]$ and then $p_1 = .6$ for time > 6.57.

We are interested in studying the properties of posted prices in the indirect mechanism. The next two propositions show the monotonicity of posted prices.

Proposition 1. *For any given group of individual goods the sequence of offered prices is decreasing.*

DOI: 10.2202/1935-1704.1659

Proof. Without loss of generality we will prove for good 1. Let $(p^1,0,t^1),...,(p^K,0,t^K)$ be the restriction of an IC DRM mechanism where the allocation involves the delivery of good one only. We order the probabilities such that $p^1 > p^2 > ... > p^K$ (which implies $t^1 > t^2 > ... > t^K$). Take the first two pieces $(p^1,0,t^1)$ and $(p^2,0,t^2)$. We know that the lowest types receive the allocation $(0,0,0)$ by the exclusion principle. We also know that for some relatively high types, $u(\theta,(p^1,0,t^1)) > 0$. Thus, there exists $\overline{\theta} > 0$ such that $p^1\overline{\theta} - t^1 = 0$.

For $(p^2,0,t^2)$ to be a relevant piece we must have $p^2\overline{\theta} - t^2 > 0$. Combining the two we get $\frac{t^1}{p^1} > \frac{t^2}{p^2}$. [3] Continuing inductively for all i we get the result. □

The next proposition states that, under a given condition, bundle prices also decrease. In the case of bundles we have that $(p_1,p_2) >> 0$ and Incentive Compatibility is not enough to guarantee decreasing prices.[4]

Proposition 2. *Let $i = 1,\ldots,M$ be the pieces of the mechanism with positive prices ordered such that the piece with $i = 1$ is delivered first, $i = 2$ second, and so on. Then if $\frac{\min\{p_1^i,p_2^i\}}{\max\{p_1^i,p_2^i\}} > \frac{\min\{p_1^{i+1},p_2^{i+1}\}}{\max\{p_1^{i+1},p_2^{i+1}\}}$ $\forall\, i = 1,...,M-1$ then the sequence of offered prices for the bundle decreases.*

Proof. Let $(p_1^1,p_2^1,t^1),...,(p_1^M,p_2^M,t^M)$ be the restriction of a IC DRM mechanism where the allocation involves delivery of both goods. We order them such that $\max\{p_1^1,p_2^1\} > ... > \max\{p_1^M,p_2^M\}$. Take the first two pieces. There exists $\overline{\theta} >> 0$ such that $u(\overline{\theta},(p_1^1,p_2^1,t^1)) = p_1^1\overline{\theta}_1 + p_2^1\overline{\theta}_2 - t^1 = 0$. Again for (p_1^2,p_2^2,t^2) to be a relevant piece, it must be the case that $u(\overline{\theta},(p_1^2,p_2^2,t^2)) = p_1^2\overline{\theta}_1 + p_2^2\overline{\theta}_2 - t^2 > 0$. Therefore $\frac{t^1}{\max\{p_1^1,p_2^1\}} = \overline{\theta}_1 + \frac{\min\{p_1^1,p_2^1\}}{\max\{p_1^1,p_2^1\}}\overline{\theta}_2 > \overline{\theta}_1 + \frac{\min\{p_1^2,p_2^2\}}{\max\{p_1^2,p_2^2\}}\overline{\theta}_2 > \frac{t^2}{\max\{p_1^2,p_2^2\}}$.

Continuing in this way we prove the property for all prices. □

We have found that whenever randomization is optimal in the DRM, the sequence of posted prices of the indirect mechanism decreases. Therefore the monopolist engages in intertemporal price discrimination. Moreover, the price of each individual good (as a function of time) is a simple decreasing function.

Finally, it is a straightforward exercise to show that we can implement all ICDRM in the manner we have described above. The proof is very similar to that of the revelation principle and it also requires to show that the expected utility a θ-type consumer obtains in the DRM is the same as the one she obtains in the indirect mechanism.[5]

[3] The expression $\frac{t^i}{\max\{p_1^i,p_2^i\}} = \frac{t^i}{p_1^i}$ is the price of good 1.

[4] A counterexample is provided in the longer version of this paper, see Runco (2010).

[5] Details can be found in Runco (2010).

3 Conclusion

We have analyzed a well-known model but with the assumption that the monopolist sells multiple goods. We have shown that stochastic DRM can easily be implemented in a deterministic way and that the monopolist will find it optimal to price discriminate as long as the optimal DRM is stochastic, providing a different rationale for IPD.

Additionally, in case of price discrimination, price functions have the very desirable property of being simple functions, in contrast with earlier results where these were continuous. Moreover, it can be proved that *any* incentive compatible stochastic direct revelation mechanism can be implemented in the manner we have described.

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DOI: 10.2202/1935-1704.1659